

Ginsparg–Wilson Relation and Spin Chains

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The Baxter 8-vertex model is equivalent to a particular lattice formulation of a self-interacting, massive Dirac fermion theory. In the time-continuum limit, the lattice Hamiltonian (XYZ spin chain) can be explicitly transformed to a lattice Dirac Hamiltonian. We show that the kernel describing the quadratic part of this Hamiltonian satisfies a one-dimensional version of the Ginsparg-Wilson relation. The corresponding conserved charge is derived and compared with the conserved arrow number of the 8-vertex model.

The continuum field theory obtained at the second-order phase transition of the 8-vertex model is equivalent to the massive Thirring model, a theory of a self-interacting Dirac fermion in two space-time dimensions [1]. This was demonstrated by examining the continuum limit of the XYZ nearest-neighbor Heisenberg spin-chain Hamiltonian, which is obtained from the 8-vertex model transfer matrix in the infinite anisotropy (i.e. time-continuum) limit. At the point where the mass is zero, the spin model reduces to a 6-vertex model (or the XXZ spin chain in the Hamiltonian limit) which exhibits a conservation of arrows at each vertex. In the spin chain Hamiltonian, this arrow conservation reduces to the conservation of the z -component of the total spin, and is a consequence of the symmetry under a global rotation of spins in the $\sigma^x - \sigma^y$ plane. Some time ago, it was argued that the conserved arrow number of the spin chain Hamiltonian corresponded to an exact lattice chiral symmetry of the equivalent fermion theory [2]. This argument relied on the transformation properties of the lattice fermion under an exact lattice Lorentz invariance [3].

Recent developments on the problem of chiral lattice fermions [4–6] have focused on the Ginsparg-Wilson (GW) relation[7]. If a lattice theory is defined by a kernel that satisfies this relation, then it has an analog of exact chiral symmetry [6]. The Nielsen-Ninomiya theorem is avoided by the fact that the chiral transformation

is not a simple on-site γ^5 rotation, but involves hopping terms present in the corresponding Dirac kernel. Here we will investigate the chiral symmetry of the 8-vertex model in the light of these new developments.

In the massless vertex model the conserved arrow charge is associated with the symmetry of the vertex Boltzmann weight under a local phase rotation of the four arrows involved in a single vertex. However, as we show here, the lattice Dirac spinor of the equivalent fermion Hamiltonian is constructed from combinations of fermionized spins of the XYZ chain, residing on different sites. Thus, the on-site phase rotation of the arrows becomes a transformation which mixes Dirac components on neighboring sites. This raises the possibility that the chiral symmetry of the vertex model is realized in a Ginsparg-Wilson form and that the conserved arrow charge is related to the charge constructed by Lüscher for theories satisfying the GW relation. Here we present some evidence to support this proposition. Since a direct transformation from the 2-dimensional vertex model to the 2-dimensional Dirac action has not been constructed yet, our analysis focuses on the symmetry of the massless spin chain.

In the present discussion, we will consider only the free fermion part of the Hamiltonian (the XY chain). After performing a Jordan-Wigner transformation which transforms the spins into canonical fermion operators, the Hamiltonian may be written as

$$H = i \sum_j c_{j+1}^x c_j^y + k c_j^x c_{j+1}^y, \quad (1)$$

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where the operators c_j^x, c_j^y satisfy

$$(c_j^a)^\dagger = c_j^a \quad \{c_j^a, c_k^b\} = \delta_{ab}\delta_{jk} \quad a, b = x, y. \quad (2)$$

The massless limit of (1) is $k \rightarrow 1$. Our aim is to put the above quadratic Hamiltonian in the standard form, involving a complex Dirac spinor. Since there are two real degrees of freedom per site, we can combine them to define a single complex canonical variable per site, namely

$$\begin{aligned} c_j &= \frac{1}{\sqrt{2}}(c_j^x + ic_{j+1}^x) & j \text{ even} \\ c_j &= \frac{1}{\sqrt{2}}(c_j^y + ic_{j+1}^y) & j \text{ odd}. \end{aligned} \quad (3)$$

By construction, these variables are canonical and satisfy

$$\{c_j^\dagger, c_k\} = \delta_{jk}, \quad (4)$$

with all other anticommutators vanishing. Defining relations (3) can be easily inverted and the Hamiltonian takes the form

$$\begin{aligned} H &= i \sum_n c_{2n} c_{2n-1}^\dagger + c_{2n}^\dagger c_{2n-1} \\ &+ k (c_{2n} c_{2n+1}^\dagger + c_{2n}^\dagger c_{2n+1}). \end{aligned} \quad (5)$$

Next, we define a Dirac spinor ψ_n living on the sublattice of the original lattice through

$$\psi_n^1 = (-1)^n c_{2n-1} \quad \psi_n^2 = (-1)^n c_{2n}, \quad (6)$$

where the factors $(-1)^n$ were introduced for later convenience. The resulting Hamiltonian on the decimated lattice then reads

$$\begin{aligned} H &= i \sum_n (\psi_n^2)^\dagger \psi_n^1 - (\psi_n^1)^\dagger \psi_n^2 \\ &- k \left((\psi_n^2)^\dagger \psi_{n+1}^1 - (\psi_{n+1}^1)^\dagger \psi_n^2 \right). \end{aligned} \quad (7)$$

Note that we could have switched the “mass” and “hopping” terms of the above Hamiltonian by shifting the indices in defining relations (6) by one.

Finally, we introduce the Dirac structure by defining

$$\gamma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (8)$$

This brings the Hamiltonian to the form

$$\begin{aligned} H &= \sum_n \bar{\psi}_n \psi_n + \frac{k}{2} \sum_n \bar{\psi}_n (\gamma_1 - 1) \psi_{n+1} \\ &- \frac{k}{2} \sum_n \bar{\psi}_n (\gamma_1 + 1) \psi_{n-1}, \end{aligned} \quad (9)$$

where $\bar{\psi} = \psi^\dagger \gamma_0$. Comparing this to the standard form of the Wilson Hamiltonian $H_W(K, M, r)$

$$\begin{aligned} H_W &= M \sum_n \bar{\psi}_n \psi_n + K \sum_n \bar{\psi}_n (\gamma_1 - r) \psi_{n+1} \\ &+ K \sum_n \bar{\psi}_n (\gamma_1 + r) \psi_{n-1}, \end{aligned} \quad (10)$$

we conclude that the Hamiltonian (1) is equivalent to $H_W(k/2, 1, 1)$, and describes a free Wilson fermion with a specific choice of Wilson parameters.

To see the connection to the GW condition and corresponding symmetry, it is convenient to work in momentum space which we define by

$$\psi_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dp \psi(p) e^{ipn}. \quad (11)$$

If we represent the Hamiltonian (9) by kernel \mathcal{H} through $H = \bar{\psi}_m \mathcal{H}_{mn} \psi_n$, then in Fourier space we have $H = 1/2\pi \int_{-\pi}^{\pi} dp \bar{\psi}(p) \mathcal{H}(p) \psi(p)$, where

$$\mathcal{H}(p) = 1 - k \cos(p) + ik \sin(p) \gamma_1. \quad (12)$$

The one-particle spectrum of this theory can be found by diagonalizing $\gamma_0 \mathcal{H}(p)$ and is given by

$$\epsilon(p)^2 = (1 - k)^2 + 2k(1 - \cos(p)). \quad (13)$$

Consequently, the spin chain Hamiltonian (1) describes a massless relativistic fermion if $k = 1$. Restricting ourselves to that value and denoting the corresponding kernel by \mathcal{H}^c , it is easy to check that

$$\mathcal{H}^c \gamma_5 + \gamma_5 \mathcal{H}^c = \mathcal{H}^c \gamma_5 \mathcal{H}^c. \quad (14)$$

In other words, \mathcal{H}^c satisfies the Ginsparg-Wilson relation.

While the above observation is quite amusing, it is not at all obvious that the Ginsparg-Wilson condition for the Hamiltonian kernel actually has any interesting symmetry consequences for the

theory as it does in the case of the Euclidean formulation [6]. To investigate this, consider some generic quadratic Hamiltonian $\bar{\psi}\mathcal{H}\psi$, on the odd-dimensional spatial lattice, such that the kernel \mathcal{H} satisfies GW relation. Let us consider the quantity

$$Q = \bar{\psi}\gamma_0\gamma_5\left(1 - \frac{1}{2}\mathcal{H}\right)\psi, \quad (15)$$

which is constructed in analogy to the one involved in the chiral transformation considered by Lüscher [6]. In the continuum limit it reduces to the standard axial charge. If we require Q to be conserved, i.e.

$$[\gamma_0\mathcal{H}, \gamma_5(1 - \frac{1}{2}\mathcal{H})] = 0, \quad (16)$$

then, in addition to GW condition, we must also impose

$$[\mathcal{H}, \gamma_0\gamma_5] = 0. \quad (17)$$

In one dimension, this additional condition reduces to $[\mathcal{H}, \gamma_1] = 0$, which is fulfilled for our \mathcal{H}^c , and Q is thus indeed conserved in this case.

It is interesting to note that using the methods of Ref. [8] it can be shown that \mathcal{H}^c is the only acceptable ultralocal solution of the GW relation in one spatial dimension, satisfying condition (17). Indeed, taking (17) into account, the most general kernel \mathcal{H} can be written in form

$$\mathcal{H}(p) = (1 - A(p))\mathbb{I} + iB(p)\gamma_1. \quad (18)$$

The GW relation then translates into the algebraic condition $A^2 + B^2 = 1$. Using a basic Lemma proved in Ref. [8], it then follows that a unique ultralocal solution giving massless relativistic spectrum is $A(p) = \cos(p)$, $B(p) = \sin(p)$. This corresponds to $\mathcal{H}^c(p)$. No ultralocal solutions in higher dimensions respecting hypercubic symmetry exist.

To see the relation of Q to the conserved arrow charge of the vertex model, it is useful to write it in terms of the spin-chain fermion operators, namely

$$Q = \frac{i}{2} \sum_j c_j^x c_j^y + c_{2j+2}^y c_{2j}^x + c_{2j-1}^y c_{2j+1}^x. \quad (19)$$

The “on-site” part is (up to proportionality constant) the arrow charge Q_A , and we denote the “next-nearest neighbour” part as Q_2 , i.e. $Q = Q_A + Q_2$. Q_2 is one of the higher conservation laws that exist in this model as a manifestation of complete integrability. The relation between Q_A and Q is somewhat reminiscent of the relation between the Hamiltonian and the log of the transfer matrix. The latter reduces to the (nearest-neighbor) Hamiltonian in the time continuum limit, but for finite lattice spacing in the time direction, the log of the transfer matrix includes higher conserved operators with higher hopping terms. It would be useful to explore these connections in the context of the full 2-dimensional vertex model, rather than being restricted to the Hamiltonian limit. An explicit construction of the 2-dimensional lattice Dirac operator \mathcal{D} for this model would be of great interest. Heuristic arguments suggest that \mathcal{D} will not be ultralocal, in accordance with the no-go theorem of Ref. [8], and so the possibility that \mathcal{D} satisfies the 2-dimensional Ginsparg-Wilson relation is not ruled out.

Let us finally note that kernel like \mathcal{H}^c was actually considered in the context of perfect fermionic actions [9]. Now, that the relation of perfect actions to Ginsparg-Wilson approach is clear, this is actually not surprising. We thank S. Chandrashekhara, P. Hasenfratz and W. Bietenholz for pointing that out to us.

REFERENCES

1. A. Luther, Phys. Rev. **B14** (1975) 2153.
2. H.B. Thacker, Nucl. Phys. **B** (Proc. Suppl.) **42** (1995) 642.
3. H.B. Thacker, Physica **18D** (1986) 348.
4. P. Hasenfratz, hep-lat/9802007.
5. H. Neuberger, Phys. Lett. **B427** (1998) 353.
6. M. Lüscher, Phys. Lett. **B428** (1998) 342.
7. P. Ginsparg and K. Wilson, Phys. Rev. **D25** (1982) 2649.
8. I. Horváth, hep-lat/9808002.
9. W. Bietenholz and U.-J. Wiese, Nucl. Phys. **B** (Proc. Suppl.) **34** (1994) 516.